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DIRECT AND INVERSE PROBLEMS FOR THERMOELASTIC PLATES. I. THE STUDY OF BENDING

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Abstract - The usefulness of plate theories resides in that they reduce complicated three-dimensional problems to simpler ones in two dimensions without compromising the essential information needed in the study of the phenomenon of bending. In this paper, problems with mixed boundary data are solved in a distributional context. Such results are necessary since they form the first stage in the study of inverse problems for this type of mechanical structures, which, in turn, play an important role in the non-destructive testing of materials.

1. INTRODUCTION

The importance of theories of elastic plates is twofold: they simplify the mathematical model by reducing it from three to two dimensions, and they focus attention on the mechanical process of bending by disregarding less important factors.

Below, we investigate the dynamic bending of a thin elastic plate subjected to external forces and moments and internal heat sources, and to homogeneous initial conditions and nonhomogeneous mixed boundary conditions. The model used is that of plates with transverse shear deformation, proposed in [1] and generalized to thermoelastic plates in [2]. The initial-boundary value problems are considered variationally, in spaces of distributions, and solved by means of the Laplace transformation technique. It is shown that the model has a unique weak solution that depends continuously on the data. The model without thermal effects was studied in [3]–[7].

The results of the direct problem are very important because they have direct bearing on the solution of the corresponding inverse problem. The latter, which plays an important role in the non-destructive testing of materials, will be investigated separately.

2. THE MATHEMATICAL PROBLEM

Suppose that a homogeneous and isotropic elastic material occupies a region $\bar{S}\times[-h_0/2,h_0/2]\subset\mathbb{R}^3$, $S\subset\mathbb{R}^2$. The displacement vector at x' and $t\geq 0$ is $v(x',t)=(v_1(x',t),v_2(x',t),v_3(x',t))^{\mathrm{T}}$, where the superscript T denotes matrix transposition. The temperature in the plate is $\theta(x',t)$. Let $x'=(x,x_3)$, $x=(x_1,x_2)\in\bar{S}$. In [1] it is assumed that

$$v(x',t)=(x_3u_1(x,t),x_3u_2(x,t),u_3(x,t))^{\mathrm{T}}.$$

When thermal effects intervene, the "averaged" temperature across thickness [2]

$$u_4(x,t)=\frac{1}{h^2h_0}\int\limits_{-h_0/2}^{h_0/2}x_3\theta(x,x_3,t)\,dx_3,\quad h^2=\frac{h_0^2}{12},$$

is also considered. Then $U(x,t)=(u(x,t)^{\mathrm{T}},u_4(x,t))^{\mathrm{T}}$, $u(x,t)=(u_1(x,t),u_2(x,t),u_3(x,t))^{\mathrm{T}}$, satisfies

$$\mathcal{B}_0\partial_t^2U(x,t)+\mathcal{B}_1\partial_tU(x,t)+\mathcal{A}U(x,t)=\mathcal{Q}(x,t),\quad (x,t)\in G=S\times(0,\infty),\tag{1}$$

where $\mathcal{B}_0=\mathrm{diag}\{\rho h^2,\rho h^2,\rho,0\}$, $\partial_t=\partial/\partial t$, $\rho>0$ is the constant density of the material,

$$\mathcal{B}_1=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\\eta\partial_1&\eta\partial_2&0&\varkappa^{-1}\end{pmatrix},\quad \mathcal{A}=\begin{pmatrix}&&&h^2\gamma\partial_1\\&A&&h^2\gamma\partial_2\\&&&0\\0&0&0&-\Delta\end{pmatrix},$$

$$A=\begin{pmatrix}-h^2\mu\Delta-h^2(\lambda+\mu)\partial_1^2+\mu&-h^2(\lambda+\mu)\partial_1\partial_2&\mu\partial_1\\-h^2(\lambda+\mu)\partial_1\partial_2&-h^2\mu\Delta-h^2(\lambda+\mu)\partial_2^2+\mu&\mu\partial_2\\-\mu\partial_1&-\mu\partial_2&-\mu\Delta\end{pmatrix},$$

$\partial_\alpha = \partial/\partial x_\alpha$, $\alpha = 1, 2$, $\eta, \varkappa$, and γ are positive physical constants, λ and μ are the Lamé coefficients of the material satisfying $\lambda + \mu > 0$, $\mu > 0$, and $\mathcal{Q}(x,t) = (q(x,t)^{\mathrm{T}}, q_4(x,t))^{\mathrm{T}}$, where $q(x,t) = (q_1(x,t), q_2(x,t), q_3(x,t))^{\mathrm{T}}$ is a combination of the forces and moments acting on the plate and its faces and $q_4(x,t)$ is a combination of the averaged heat source density and the temperature and heat flux on the faces.

For simplicity, suppose that

$$U(x,0) = 0, \quad \partial_t u(x,0) = 0, \quad x \in S, \tag{2}$$

which is no restriction on generality since nonhomogeneous initial conditions can easily be made homogeneous (see [8]).

We assume that the boundary ∂S of S is a simple, closed, piecewise smooth curve that consists of four open arcs counted counterclockwise as ∂S_i, $i = 1, \dots, 4$, such that

$$\partial S = \bigcup_{i=1}^{4} \overline{\partial S_i}, \quad \partial S_i \cap \partial S_j = \emptyset,\ i \neq j, \quad i, j = 1, \dots, 4.$$

For $i, j = 1, 2, 3, 4$, we write

$$\Gamma = \partial S \times (0, \infty), \quad \Gamma_i = \partial S_i \times (0, \infty),$$
$$\partial S_{ij} = \partial S_i \cup \partial S_j \cup (\overline{\partial S_i} \cap \overline{\partial S_j}), \quad \Gamma_{ij} = \partial S_{ij} \times (0, \infty).$$

Let

$$u(x,t) = f(x,t), \quad u_4(x,t) = f_4(x,t), \quad (x,t) \in \Gamma_1, \tag{3}$$
$$u(x,t) = f(x,t), \quad \partial_n u_4(x,t) = g_4(x,t), \quad (x,t) \in \Gamma_2, \tag{4}$$

where $n = n(x) = (n_1(x), n_2(x), n_3(x))^{\mathrm{T}}$ is the outward unit normal to ∂S and $\partial_n = \partial/\partial n$.

The moment-force boundary operator [1] is

$$T = \begin{pmatrix} h^2\big[(\lambda + 2\mu) n_1 \partial_1 + \mu n_2 \partial_2\big] & h^2(\lambda n_1 \partial_2 + \mu n_2 \partial_1) & 0 \\ h^2(\mu n_1 \partial_2 + \lambda n_2 \partial_1) & h^2\big[(\lambda + 2\mu) n_2 \partial_2 + \mu n_1 \partial_1\big] & 0 \\ \mu n_1 & \mu n_2 & \mu \partial_n \end{pmatrix}.$$

We also assume that

$$Tu(x,t) - h^2\gamma n(x) u_4(x,t) = g(x,t), \quad \partial_n u_4(x,t) = g_4(x,t), \quad (x,t) \in \Gamma_3, \tag{5}$$
$$Tu(x,t) - h^2\gamma n(x) u_4(x,t) = g(x,t), \quad u_4(x,t) = f_4(x,t), \quad (x,t) \in \Gamma_4. \tag{6}$$

All $f(x,t)$, $f_4(x,t)$, $g(x,t)$, and $g_4(x,t)$ in (3)–(6) are given functions.

We denote by S^+ and S^- the interior and exterior domains bounded by ∂S, respectively, and write $G^\pm = S^\pm \times (0, \infty)$. The interior and exterior initial-boundary value problems (TM$^\pm$) consist in finding $U \in \mathrm{C}^2(G^\pm) \cap \mathrm{C}^1(\bar{G}^\pm)$ that satisfies (1) in $G^\pm$, (2) in $S^\pm$, and (3)–(6).

3. THE LAPLACE-TRANSFORMED PROBLEMS (TM$_p^\pm$)

Let the Laplace transform of a function $s(x,t)$ be

$$\hat{s}(x,p) = (\mathcal{L}s)(x,p) = \int_0^\infty e^{-pt} s(x,t)\, dt.$$

Applying the Laplace transformation in (TM$^\pm$) yields elliptic boundary value problems (TM$_p^\pm$) that depend on the complex parameter p and consist in finding $\hat{U} \in \mathrm{C}^2(S^\pm) \cap \mathrm{C}^1(\bar{S}^\pm)$ that satisfies

$$p^2 \mathcal{B}_0 \hat{U}(x,p) + p\, \mathcal{B}_1 \hat{U}(x,p) + \mathcal{A}\hat{U}(x,p) = \hat{\mathcal{Q}}(x,p), \quad x \in S^\pm, \tag{7}$$

and

$$\hat{u}(x,p) = \hat{f}(x,p), \quad \hat{u}_4(x,p) = \hat{f}_4(x,p), \quad x \in \partial S_1,$$
$$\hat{u}(x,p) = \hat{f}(x,p), \quad \partial_n \hat{u}_4(x,p) = \hat{g}_4(x,p), \quad x \in \partial S_2,$$
$$T\hat{u}(x,p) - h^2\gamma n(x) \hat{u}_4(x,p) = \hat{g}(x,p), \quad \partial_n \hat{u}_4(x,p) = \hat{g}_4(x,p), \quad x \in \partial S_3,$$
$$T\hat{u}(x,p) - h^2\gamma n(x) \hat{u}_4(x,p) = \hat{g}(x,p), \quad \hat{u}_4(x,p) = \hat{f}_4(x,p), \quad x \in \partial S_4.$$

Let $m \in \mathbb{R}$ and $p \in \mathbb{C}$. We introduce the following function spaces.

• $H_m(\mathbb{R}^2)$: the Sobolev space of functions $\hat{v}_4(x)$ with norm $\|\hat{v}_4\|_m = \left\{ \int_{\mathbb{R}^2} (1+|\xi|^2)^m |\tilde{v}_4(\xi)|^2 \, d\xi \right\}^{1/2}$, where $\tilde{v}_4(\xi)$ is the Fourier transform of $\hat{v}_4(x)$.

• $H_{m,p}(\mathbb{R}^2)$: the space of vector functions $\hat{v}(x)$ which coincides with $[H_m(\mathbb{R}^2)]^3$ as a set but is endowed with the norm $\|\hat{v}\|_{m,p} = \left\{ \int_{\mathbb{R}^2} (1+|\xi|^2+|p|^2)^m |\tilde{v}(\xi)|^2 \, d\xi \right\}^{1/2}$.

• $H_m(S^\pm)$, $H_{m,p}(S^\pm)$: the spaces of the restrictions to $S^\pm$ of all $\hat{v}_4 \in H_m(\mathbb{R}^2)$ and $\hat{v} \in H_{m,p}(\mathbb{R}^2)$, respectively, with norms $\|\hat{u}_4\|_{m;S^\pm} = \inf_{\hat{v}_4 \in H_m(\mathbb{R}^2):\, \hat{v}_4|_{S^\pm} = \hat{u}_4} \|\hat{v}_4\|_m$ and $\|\hat{u}\|_{m,p;S^\pm} = \inf_{\hat{v} \in H_{m,p}(\mathbb{R}^2):\, \hat{v}|_{S^\pm} = \hat{u}} \|\hat{v}\|_{m,p}$.

• $\mathring{H}_m(S^\pm)$, $\mathring{H}_{m,p}(S^\pm)$: the subspaces of $H_m(\mathbb{R}^2)$ and $H_{m,p}(\mathbb{R}^2)$ of all $\hat{v}_4 \in H_m(\mathbb{R}^2)$ and $\hat{v} \in H_{m,p}(\mathbb{R}^2)$ with $\operatorname{supp} \hat{v}_4 \subset \bar{S}^\pm$ and $\operatorname{supp} \hat{v} \subset \bar{S}^\pm$, respectively; the norms on them are those induced by $\|\hat{v}_4\|_m$ and $\|\hat{v}\|_{m,p}$, so we denote them by the same symbols.

• $H_{-m}(S^\pm)$, $H_{-m,p}(S^\pm)$: the duals of $\mathring{H}_m(S^\pm)$, $\mathring{H}_{m,p}(S^\pm)$ with respect to the duality generated by the inner products in $L^2(S^\pm)$ and $\left[L^2(S^\pm)\right]^3$.

• $H_{1/2}(\partial S)$, $H_{1/2,p}(\partial S)$: the spaces of the traces on ∂S of all $\hat{u}_4 \in H_1(S^+)$ and $\hat{u} \in H_{1,p}(S^+)$, equipped with the norms $\|\hat{f}_4\|_{1/2;\partial S} = \inf_{\hat{u}_4 \in H_1(S^+):\, \hat{u}_4|_{\partial S} = \hat{f}_4} \|\hat{u}_4\|_{1;S^+}$ and $\|\hat{f}\|_{1/2,p;\partial S} = \inf_{\hat{u} \in H_{1,p}(S^+):\, \hat{u}|_{\partial S} = \hat{f}} \|\hat{u}\|_{1,p;S^+}$.

• $H_{-1/2}(\partial S)$, $H_{-1/2,p}(\partial S)$: the duals of $H_{1/2}(\partial S)$ and $H_{1/2,p}(\partial S)$ with respect to the duality generated by the inner products in $L^2(\partial S)$ and $[L^2(\partial S)]^3$; their norms are denoted by $\|\hat{g}_4\|_{-1/2,\partial S}$ and $\|\hat{g}\|_{-1/2,p;\partial S}$.

Let $\gamma^\pm$ be the continuous (uniformly with respect to $p \in \mathbb{C}$) trace operators from $H_1(S^\pm)$ to $H_{1/2}(\partial S)$ and from $H_{1,p}(S^\pm)$ to $H_{1/2,p}(\partial S)$.

We denote by $\partial\tilde{S} \subset \partial S$ any open part of ∂S with $\operatorname{mes} \partial\tilde{S} > 0$, and by $\tilde{\pi}$ the operator of restriction from ∂S to $\partial\tilde{S}$. We list a few more necessary spaces.

• $H_{\pm 1/2}(\partial\tilde{S})$, $H_{\pm 1/2,p}(\partial\tilde{S})$: the spaces of the restrictions to $\partial\tilde{S}$ of all the elements of $H_{\pm 1/2}(\partial S)$ and $H_{\pm 1/2,p}(\partial S)$, respectively, with norms $\|\hat{e}_4\|_{\pm 1/2;\partial\tilde{S}} = \inf_{\hat{r}_4 \in H_{\pm 1/2}(\partial S):\, \tilde{\pi}\hat{r}_4 = \hat{e}_4} \|\hat{r}_4\|_{\pm 1/2;\partial S}$ and $\|\hat{e}\|_{\pm 1/2,p;\partial\tilde{S}} = \inf_{\hat{r} \in H_{\pm 1/2,p}(\partial S):\, \tilde{\pi}\hat{r} = \hat{e}} \|\hat{r}\|_{\pm 1/2,p;\partial S}$.

• $\mathring{H}_{\pm 1/2}(\partial\tilde{S})$, $\mathring{H}_{\pm 1/2,p}(\partial\tilde{S})$: the subspaces of $H_{\pm 1/2}(\partial S)$ and $H_{\pm 1/2,p}(\partial S)$ consisting of all the elements with support in $\overline{\partial\tilde{S}}$; the norms on $\hat{e}_4 \in \mathring{H}_{\pm 1/2}(\partial\tilde{S})$ and $\hat{e} \in \mathring{H}_{\pm 1/2}(\partial\tilde{S})$ may be denoted by $\|\hat{e}_4\|_{\pm 1/2,\partial S}$ and $\|\hat{e}\|_{\pm 1/2,p;\partial S}$. We remark that $H_{\pm 1/2}(\partial\tilde{S})$ are the duals of $\mathring{H}_{\mp 1/2}(\partial\tilde{S})$ and $H_{\pm 1/2,p}(\partial\tilde{S})$ are the duals of $\mathring{H}_{\mp 1/2,p}(\partial\tilde{S})$ with respect to the duality generated by the inner products in $L^2(\partial\tilde{S})$ and $[L^2(\partial\tilde{S})]^3$.

Let π_i and π_{ij}, $i, j = 1, \dots, 4$, be the operators of restriction from ∂S to ∂S_i and from ∂S to ∂S_{ij}. Still more spaces are defined below.

• $H_1(S^\pm, \partial S_{23})$, $H_{1,p}(S^\pm, \partial S_{34})$: the subspaces of $H_1(S^\pm)$ and $H_{1,p}(S^\pm)$ consisting of all $\hat{u}_4 \in H_1(S^\pm)$ and $\hat{u} \in H_{1,p}(S^\pm)$ such that $\pi_{41}\gamma^\pm \hat{u}_4 = 0$ and $\pi_{12}\gamma^\pm \hat{u} = 0$.

• $H_{-1}(S^\pm, \partial S_{23})$, $H_{-1,p}(S^\pm, \partial S_{34})$: the duals of $H_1(S^\pm, \partial S_{23})$ and $H_{1,p}(S^\pm, \partial S_{34})$ with respect to the original dualities; the norms of $\hat{q}_4 \in H_{-1}(S^\pm, \partial S_{23})$ and $\hat{q} \in H_{-1,p}(S^\pm, \partial S_{34})$ are denoted by $[\hat{q}_4]_{-1;S^\pm,\partial S_{23}}$ and $[\hat{q}]_{-1,p;S^\pm,\partial S_{34}}$.

• $\mathcal{H}_{1,p}(S^\pm) = H_{1,p}(S^\pm) \times H_1(S^\pm)$, with the norm of its elements $\hat{U} = (\hat{u}^{\mathrm{T}}, \hat{u}_4)^{\mathrm{T}}$ defined by $|||\hat{U}|||_{1,p;S^\pm} = \|\hat{u}\|_{1,p;S^\pm} + \|\hat{u}_4\|_{1;S^\pm}$.

• $\mathcal{H}_{1,p}(S^\pm; \partial S_{34}, \partial S_{23}) = H_{1,p}(S^\pm, \partial S_{34}) \times H_1(S^\pm, \partial S_{23})$, a subspace of $\mathcal{H}_{1,p}(S^\pm)$.

Let $\kappa > 0$, and let $\mathbb{C}_\kappa = \{p = \sigma + i\tau \in \mathbb{C} : \sigma > \kappa\}$. Below, c stands for all positive constants occurring in estimates which are independent of the functions in those estimates and of $p \in \mathbb{C}_\kappa$, but may depend on κ. Also, we denote by $(\cdot\,,\cdot)_{0;S^\pm}$, $(\cdot\,,\cdot)_{0;\partial S}$, and $(\cdot\,,\cdot)_{0;\partial\tilde{S}}$ the inner products in $[L^2(S^\pm)]^m$, $[L^2(\partial S)]^m$, and $[L^2(\partial\tilde{S})]^m$, respectively, for all $m \in \mathbb{N}$, and by $\|\cdot\|_{0;S^\pm}$, $\|\cdot\|_{0;\partial S}$, and $\|\cdot\|_{0;\partial\tilde{S}}$ the norms on the same spaces.

Let $\hat{U}(x,p) = (\hat{u}(x,p)^{\mathrm{T}}, \hat{u}_4(x,p))^{\mathrm{T}}$ be the classical solution of either problem ($\mathrm{TM}_p^\pm$), of class $\mathrm{C}^2(S^\pm) \cap \mathrm{C}^1(\bar{S}^\pm)$. We choose any function (with compact support in the case of S^-) $\hat{W}(x,p) = (\hat{w}(x,p)^{\mathrm{T}}, \hat{w}_4(x,p))^{\mathrm{T}}$, $\hat{W} \in \mathrm{C}_0^\infty(\bar{S}^\pm)$, such that $\hat{w}(x,p) = 0$ for $x \in \partial S_{12}$ and $\hat{w}_4(x,p) = 0$ for $x \in \partial S_{41}$, and multiply (7) by $\hat{W}$ in $[L^2(S^\pm)]^4$ to arrive at the equation

$$\Upsilon_{\pm,p}(\hat{U}, \hat{W}) = (\hat{\mathcal{Q}}, \hat{W})_{0;S^\pm} \pm L(\hat{W}), \tag{8}$$

where

$$\Upsilon_{\pm,p}(\hat{U}, \hat{W}) = a_\pm(\hat{u}, \hat{w}) + (\nabla\hat{u}_4, \nabla\hat{w}_4)_{0;S^\pm} + p^2(B_0^{1/2}\hat{u}, B_0^{1/2}\hat{w})_{0;S^\pm}$$
$$+ \varkappa^{-1}p(\hat{u}_4, \hat{w}_4)_{0;S^\pm} - h^2\gamma(\hat{u}_4, \operatorname{div}\hat{w})_{0;S^\pm} + \eta p(\operatorname{div}\hat{u}, \hat{w}_4)_{0;S^\pm},$$

$$a_\pm(\hat u,\hat w)=2\int\limits_{S^\pm}E(\hat u,\hat w)\,dx,$$

$$2E(\hat u,\hat w)=h^2E_0(\hat u,\hat w)+h^2\mu(\partial_2\hat u_1+\partial_1\hat u_2)(\partial_2\bar{\hat w}_1+\partial_1\bar{\hat w}_2)$$
$$+\mu\big[(\hat u_1+\partial_1\hat u_3)(\bar{\hat w}_1+\partial_1\bar{\hat w}_3)+(\hat u_2+\partial_2\hat u_3)(\bar{\hat w}_2+\partial_2\bar{\hat w}_3)\big],$$
$$E_0(\hat u,\hat w)=(\lambda+2\mu)\big[(\partial_1\hat u_1)(\partial_1\bar{\hat w}_1)+(\partial_2\hat u_2)(\partial_2\bar{\hat w}_2)\big]+\lambda\big[(\partial_1\hat u_1)(\partial_2\bar{\hat w}_2)+(\partial_2\hat u_2)(\partial_1\bar{\hat w}_1)\big],$$
$$B_0=\mathrm{diag}\{\rho h^2,\rho h^2,\rho\},\quad L(\hat W)=(\hat g_4,\hat w_4)_{0;\partial S_{23}}+(\hat g,\hat w)_{0;\partial S_{34}}.$$

Consequently, the variational problems $(\mathrm{TM}_p^\pm)$ consist in finding $\hat U\in\mathcal{H}_{1,p}(S^\pm)$ that satisfies (8) for any $\hat W\in\mathcal{H}_{1,p}(S^\pm;\partial S_{34},\partial S_{23})$ and

$$\pi_{12}\gamma^\pm\hat u=\hat f,\quad \pi_{41}\gamma^\pm\hat u_4=\hat f_4.$$

Theorem 1. *For all $\hat q\in H_{-1,p}(S^\pm,\partial S_{34})$, $\hat q_4\in H_{-1}(S^\pm,\partial S_{23})$, $\hat f\in H_{1/2,p}(\partial S_{12})$, $\hat f_4\in H_{1/2}(\partial S_{41})$, $\hat g\in H_{-1/2,p}(\partial S_{34})$, and $\hat g_4\in H_{-1/2}(\partial S_{23})$, $p\in\mathbb{C}_\kappa$, $\kappa>0$, problems $(\mathrm{TM}_p^\pm)$ have unique solutions $\hat U(x,p)\in\mathcal{H}_{1,p}(S^\pm)$, which satisfy*

$$|||\hat U|||_{1,p;S^\pm}\le c\big\{|p|[\hat q]_{-1,p;S^\pm,\partial S_{34}}+[\hat q_4]_{-1;S^\pm,\partial S_{23}}+|p|\big(\|\hat f\|_{1/2,p;\partial S_{12}}+\|\hat f_4\|_{1/2,\partial S_{41}}\big)$$
$$+|p|\|\hat g\|_{-1/2,p;\partial S_{34}}+\|\hat g_4\|_{-1/2,\partial S_{23}}\big\}.$$

To perform our full analysis, we need to introduce a few more function spaces. Let $\partial\tilde S\subset\partial S$, $\kappa>0$, and $k\in\mathbb{R}$.

- $H_{\pm1/2}(\partial\tilde S)$, $H_1(S^\pm)$, $H_{-1}(S^\pm,\partial S_{34})$: the spaces $H_{\pm1/2,p}(\partial\tilde S)$, $H_{1,p}(S^\pm)$, and $H_{-1,p}(S^\pm,\partial S_{34})$ with $p=0$; their norms are denoted by $\|\cdot\|_{\pm1/2;\partial\tilde S}$, $\|\cdot\|_{1;S^\pm}$, and $[\cdot]_{-1;S^\pm,\partial S_{34}}$.
- $H^{\mathcal L}_{\pm1/2,k,\kappa}(\partial\tilde S)$, $H^{\mathcal L}_{1,k,\kappa}(S^\pm)$, $H^{\mathcal L}_{-1,k,\kappa}(S^\pm,\partial S_{34})$: the spaces of all vector-functions $\hat e(x,p)$, $\hat u(x,p)$, and $\hat q(x,p)$ that define holomorphic mappings $\hat e(x,p):\mathbb{C}_\kappa\mapsto H_{\pm1/2}(\partial\tilde S)$, $\hat u(x,p):\mathbb{C}_\kappa\mapsto H_1(S^\pm)$, and $\hat q(x,p):\mathbb{C}_\kappa\mapsto H_{-1}(S^\pm,\partial S_{34})$, and for which

$$\|\hat e\|^2_{\pm1/2,k,\kappa;\partial\tilde S}=\sup_{\sigma>\kappa}\int\limits_{-\infty}^{\infty}(1+|p|^2)^k\|\hat e(x,p)\|^2_{\pm1/2,p;\partial\tilde S}\,d\tau<\infty,$$
$$\|\hat u\|^2_{1,k,\kappa;S^\pm}=\sup_{\sigma>\kappa}\int\limits_{-\infty}^{\infty}(1+|p|^2)^k\|\hat u(x,p)\|^2_{1,p;S^\pm}\,d\tau<\infty,$$
$$[\hat q]^2_{-1,k,\kappa;S^\pm,\partial S_{34}}=\sup_{\sigma>\kappa}\int\limits_{-\infty}^{\infty}(1+|p|^2)^k[\hat q(x,p)]^2_{-1,p;S^\pm,\partial S_{34}}\,d\tau<\infty.$$

- $H^{\mathcal L}_{\pm1/2,k,\kappa}(\partial\tilde S)$, $H^{\mathcal L}_{1,k,\kappa}(S^\pm)$, and $H^{\mathcal L}_{-1,k,\kappa}(S^\pm,\partial S_{23})$: the spaces of all $\hat e_4(x,p)$, $\hat u_4(x,p)$, and $\hat q_4(x,p)$ that define holomorphic mappings $\hat e_4(x,p):\mathbb{C}_\kappa\mapsto H_{\pm1/2}(\partial\tilde S)$, $\hat u_4(x,p):\mathbb{C}_\kappa\mapsto H_1(S^\pm)$, and $\hat q_4(x,p):\mathbb{C}_\kappa\mapsto H_{-1}(S^\pm,\partial S_{23})$, and for which

$$\|\hat e_4\|^2_{\pm1/2,k,\kappa;\partial\tilde S}=\sup_{\sigma>\kappa}\int\limits_{-\infty}^{\infty}(1+|p|^2)^k\|\hat e_4(x,p)\|^2_{\pm1/2,\partial\tilde S}\,d\tau<\infty,$$
$$\|\hat u_4\|^2_{1,k,\kappa;S^\pm}=\sup_{\sigma>\kappa}\int\limits_{-\infty}^{\infty}(1+|p|^2)^k\|\hat u_4(x,p)\|^2_{1;S^\pm}\,d\tau<\infty,$$
$$[\hat q_4]^2_{-1,k,\kappa;S^\pm,\partial S_{23}}=\sup_{\sigma>\kappa}\int\limits_{-\infty}^{\infty}(1+|p|^2)^k[\hat q_4(x,p)]^2_{-1;S^\pm,\partial S_{23}}\,d\tau<\infty.$$

- $\mathcal{H}_1(S^\pm)=H_1(S^\pm)\times H_1(S^\pm)$, with norms $|||\hat U|||_{1;S^\pm}=\|\hat u\|_{1;S^\pm}+\|\hat u_4\|_{1;S^\pm}$.
- $\mathcal{H}^{\mathcal L}_{1,k,l,\kappa}(S^\pm)=H^{\mathcal L}_{1,k,\kappa}(S^\pm)\times H^{\mathcal L}_{1,l,\kappa}(S^\pm)$, with norms $|||\hat U|||_{1,k,l,\kappa;S^\pm}=\|\hat u\|_{1,k,\kappa;S^\pm}+\|\hat u_4\|_{1,l,\kappa;S^\pm}$.

Theorem 2. *Let $\kappa > 0$ and $l \in \mathbb{R}$. If*

$$\hat{q}(x,p) \in H^{\mathcal{L}}_{-1,l+1,\kappa}(S^{\pm}, \partial S_{34}), \quad \hat{q}_4(x,p) \in H^{\mathcal{L}}_{-1,l,\kappa}(S^{\pm}, \partial S_{23}),$$
$$\hat{f}(x,p) \in H^{\mathcal{L}}_{1/2,l+1,\kappa}(\partial S_{12}), \quad \hat{f}_4(x,p) \in H^{\mathcal{L}}_{1/2,l+1,\kappa}(\partial S_{41}),$$
$$\hat{g}(x,p) \in H^{\mathcal{L}}_{-1/2,l+1,\kappa}(\partial S_{34}), \quad \hat{g}_4(x,p) \in H^{\mathcal{L}}_{-1/2,l,\kappa}(\partial S_{23}),$$

then the (*weak*) *solutions* $\hat{U}(x,p) = (\hat{u}(x,p)^{\mathrm{T}}, \hat{u}_4(x,p))^{\mathrm{T}}$ *of problems* $(\mathrm{TM}^{\pm}_p)$ *belong to* $\mathcal{H}^{\mathcal{L}}_{1,l,l,\kappa}(S^{\pm})$ *and*

$$|||\hat{U}|||_{1,l,l,\kappa;S^{\pm}} \le c\big\{[\hat{q}]_{-1,l+1,\kappa;S^{\pm},\partial S_{34}} + [\hat{q}_4]_{-1,l,\kappa;S^{\pm},\partial S_{23}} + \|\hat{f}\|_{1/2,l+1,\kappa;\partial S_{12}} + \|\hat{f}_4\|_{1/2,l+1,\kappa;\partial S_{41}} + \|\hat{g}\|_{-1/2,l+1,\kappa;\partial S_{34}} + \|\hat{g}_4\|_{-1/2,l,\kappa;\partial S_{23}}\big\}.$$

4. EXISTENCE AND UNIQUENESS OF SOLUTIONS

We need to define one more set of functions spaces. Let $\kappa > 0$ and $k, l \in \mathbb{R}$. By

$$H^{\mathcal{L}^{-1}}_{1,k,\kappa}(G^{\pm}), \quad H^{\mathcal{L}^{-1}}_{1,l,\kappa}(G^{\pm}), \quad \mathcal{H}^{\mathcal{L}^{-1}}_{1,k,l,\kappa}(G^{\pm}) = H^{\mathcal{L}^{-1}}_{1,k,\kappa}(G^{\pm}) \times H^{\mathcal{L}^{-1}}_{1,l,\kappa}(G^{\pm}),$$
$$H^{\mathcal{L}^{-1}}_{-1,l,\kappa}(G^{\pm}, \Gamma_{34}), \quad H^{\mathcal{L}^{-1}}_{-1,l,\kappa}(G^{\pm}, \Gamma_{23}), \quad H^{\mathcal{L}^{-1}}_{1/2,l,\kappa}(\Gamma_{12}),$$
$$H^{\mathcal{L}^{-1}}_{1/2,l,\kappa}(\Gamma_{41}), \quad H^{\mathcal{L}^{-1}}_{-1/2,l,\kappa}(\Gamma_{34}), \quad H^{\mathcal{L}^{-1}}_{-1/2,l,\kappa}(\Gamma_{23})$$

we denote the spaces consisting of the inverse Laplace transforms of the elements of

$$H^{\mathcal{L}}_{1,k,\kappa}(S^{\pm}), \quad H^{\mathcal{L}}_{1,l,\kappa}(S^{\pm}), \quad \mathcal{H}^{\mathcal{L}}_{1,k,l,\kappa}(S^{\pm}) = H^{\mathcal{L}}_{1,k,\kappa}(S^{\pm}) \times H^{\mathcal{L}}_{1,l,\kappa}(S^{\pm}),$$
$$H^{\mathcal{L}}_{-1,l,\kappa}(S^{\pm}, \partial S_{34}), \quad H^{\mathcal{L}}_{-1,l,\kappa}(S^{\pm}, \partial S_{23}), \quad H^{\mathcal{L}}_{1/2,l,\kappa}(\partial S_{12}),$$
$$H^{\mathcal{L}}_{1/2,l,\kappa}(\partial S_{41}), \quad H^{\mathcal{L}}_{-1/2,l,\kappa}(\partial S_{34}), \quad H^{\mathcal{L}}_{-1/2,l,\kappa}(\partial S_{23}),$$

respectively. The norms on these spaces are defined by

$$\|u\|_{1,k,\kappa;G^{\pm}} = \|\hat{u}\|_{1,k,\kappa;S^{\pm}}, \quad \|u_4\|_{1,l,\kappa;G^{\pm}} = \|\hat{u}_4\|_{1,l,\kappa;S^{\pm}},$$
$$|||U|||_{1,k,l,\kappa;G^{\pm}} = |||\hat{U}|||_{1,k,l,\kappa;S^{\pm}},$$
$$[q]_{-1,l,\kappa;G^{\pm},\Gamma_{34}} = [\hat{q}]_{1,l,\kappa;S^{\pm},\partial S_{34}}, \quad [q_4]_{-1,l,\kappa;G^{\pm},\Gamma_{23}} = [\hat{q}_4]_{1,l,\kappa;S^{\pm},\partial S_{23}},$$
$$\|f\|_{1/2,l,\kappa;\Gamma_{12}} = \|\hat{f}\|_{1/2,l,\kappa;\partial S_{12}}, \quad \|f_4\|_{1/2,l,\kappa;\Gamma_{41}} = \|\hat{f}_4\|_{1/2,l,\kappa;\partial S_{41}},$$
$$\|g\|_{-1/2,l,\kappa;\Gamma_{34}} = \|\hat{g}\|_{-1/2,l,\kappa;\partial S_{34}}, \quad \|g_4\|_{-1/2,l,\kappa;\Gamma_{23}} = \|\hat{g}_4\|_{-1/2,l,\kappa;\partial S_{23}}.$$

For simplicity, we also use $\gamma^{\pm}$ to denote the trace operators from $G^{\pm}$ to Γ, and π_{ij} to denote the operators of restriction from Γ to its parts Γ_{ij}, $i, j = 1, 2, 3, 4$.

$U \in \mathcal{H}^{\mathcal{L}^{-1}}_{1,0,0,\kappa}(G^{\pm})$, $U(x,t) = (u(x,t)^{\mathrm{T}}, u_4(x,t))^{\mathrm{T}}$, is called a weak solution of $(\mathrm{TM}^{\pm})$ if

(i) $\gamma_0 u = 0$, where γ_0 is the trace operator on $S^{\pm} \times \{t = 0\}$;

(ii) $\pi_{12}\gamma^{\pm}u = f(x,t)$ and $\pi_{41}\gamma^{\pm}u_4 = f_4(x,t)$;

(iii) U satisfies

$$\Upsilon_{\pm}(U,W) = \int_0^{\infty} (\mathcal{Q}, W)_{0;S^{\pm}}\, dt \pm L(W),$$

where

$$\Upsilon_{\pm}(U,W) = \int_0^{\infty} \big\{a_{\pm}(u,w) + (\nabla u_4, \nabla w_4)_{0;S^{\pm}} - (B_0^{1/2}\partial_t u, B_0^{1/2}\partial_t w)_{0;S^{\pm}} - \varkappa^{-1}(u_4, \partial_t w_4)_{0;S^{\pm}} - h^2\gamma(u_4, \operatorname{div} w)_{0;S^{\pm}} - \eta(\operatorname{div} u, \partial_t w_4)_{0;S^{\pm}}\big\}\, dt,$$
$$L(W) = \int_0^{\infty} \big\{(g, w)_{0;\partial S_{34}} + (g_4, w_4)_{0;\partial S_{23}}\big\}\, dt,$$

for all $W \in \mathrm{C}_0^{\infty}(\bar{G}^{\pm})$, $W(x,t) = (w(x,t)^{\mathrm{T}}, w_4(x,t))^{\mathrm{T}}$, such that $w(x,t) = 0$ for $(x,t) \in \Gamma_{12}$ and $w_4(x,t) = 0$ for $(x,t) \in \Gamma_{41}$.

Theorem 3. *Let $U(x,t)=\mathcal{L}^{-1}\hat{U}(x,p)$ be the inverse Laplace transform of the weak solution $\hat{U}(x,p)$ of either of the problems* $(\mathrm{TM}_p^{\pm})$*. If*

$$q(x,t)\in H^{\mathcal{L}^{-1}}_{-1,l+1,\kappa}(G^{\pm},\Gamma_{34}),\quad q_4(x,t)\in H^{\mathcal{L}^{-1}}_{-1,l,\kappa}(G^{\pm},\Gamma_{23}),$$
$$f(x,t)\in H^{\mathcal{L}^{-1}}_{1/2,l+1,\kappa}(\Gamma_{12}),\quad f_4(x,t)\in H^{\mathcal{L}^{-1}}_{1/2,l+1,\kappa}(\Gamma_{41}),$$
$$g(x,t)\in H^{\mathcal{L}^{-1}}_{-1/2,l+1,\kappa}(\Gamma_{34}),\quad g_4(x,t)\in H^{\mathcal{L}^{-1}}_{-1/2,l,\kappa}(\Gamma_{23}),$$

where $\kappa>0$ and $l\in\mathbb{R}$, then $U\in\mathcal{H}^{\mathcal{L}^{-1}}_{1,l,l,\kappa}(G^{\pm})$ and

$$|||U|||_{1,l,l,\kappa;G^{\pm}}\le c\big\{[q]_{-1,l+1,\kappa;G^{\pm},\Gamma_{34}}+[q_4]_{-1,l,\kappa;G^{\pm},\Gamma_{23}}+\|f\|_{1/2,l+1,\kappa;\Gamma_{12}}+\|f_4\|_{1/2,l+1,\kappa;\Gamma_{41}}$$
$$+\|g\|_{-1/2,l+1,\kappa;\Gamma_{34}}+\|g_4\|_{-1/2,l,\kappa;\Gamma_{23}}\big\}.$$

If, in addition, $l\ge 0$, then U is a weak solution of the corresponding problem $(\mathrm{TM}^{\pm})$.

Theorem 4. *Each of the problems* $(\mathrm{TM}^{\pm})$ *has at most one weak solution.*

Full details of the proofs of Theorems 1–4 will be published elsewhere.

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